SECTION 5.4: WORKING WITH INTEGRALS

RECALL: INTEGRALS AND SYMMETRY

Suppose f is continuous. Then:

- If f is **even**, that is if $f(-x) = f(x)$ for all x, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is **odd**, that is if $f(-x) = -f(x)$ for all x, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.
- Integrals of sine and cosine functions over a complete period is 0.

EXAMPLE 1: Use symmetry to help you find the following definite integrals:

1. $\displaystyle\int_{-1}^{1} 3x^4\,dx$

   For $f(x) = 3x^4$, $f(-x) = 3(-x)^4 = 3x^4 = f(x)$, so f is even.

   Hence, $\displaystyle\int_{-1}^{1} 3x^4\,dx = 2\int_{0}^{1} 3x^4 = \frac{6}{5}x^5\Big|_{x=0}^{x=1} = \frac{6}{5}$.

2. $\displaystyle\int_{-\pi}^{\pi} 2x\cos^3(x)\,dx$

   For $f(x) = 2x\cos^3(x) = 2x\,(\cos(x))^3$, $f(-x) = -2x(\cos(-x))^3 = -2x(\cos(x))^3 = -f(x)$, so f is odd.

   Hence, $\displaystyle\int_{-1}^{1} 2x\cos^3(x)\,dx = 0$.

3. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(8t)\,dt$

   The period of $f(t) = \sin(8t)$ is $\dfrac{2\pi}{8} = \dfrac{\pi}{4}$.

   Since $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$ has a length of $\dfrac{3\pi}{4} - \dfrac{\pi}{4} = \dfrac{\pi}{2}$, $f(t)$ makes two complete cycles over this interval.

   Hence, $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(8t)\,dt = 0$.

THE AVERAGE VALUE OF A FUNCTION

Suppose f is continuous on $[a, b]$ and we wish to find the 'average' value of f, which we denote $\overline{f}$. We could start by finding the average of f at the (right) endpoints of a regular partition: $\{x_0, x_1, x_2, \dots x_n\}$ in $[a, b]$:

$$\overline{f} \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

Since the points x_i are equally distributed, we have $\Delta x_i = \dfrac{b-a}{n}$ so that $\dfrac{\Delta x_i}{b-a} = \dfrac{1}{n}$. Hence:

$$\overline{f} \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{\Delta x_i}{b-a}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x_i$$

To get a better and better approximation, we let $n \to \infty$. We recognize the sum on the right as a Riemann Sum.

Hence, we **define** the average value of f as:

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

EXAMPLE 2: The temperature $T(t)$ (in degrees Fahrenheit) t hours after 6 AM is given by:

$$T(t) = -0.5t^2 + 5t + 72.5, \quad 0 \le t \le 10.$$

1. Find the average temperature between 8 AM and noon rounded to the nearest degree.

   Since 8 AM corresponds to $t = 2$ and noon corresponds to $t = 6$, we compute:

$$\overline{T} = \frac{1}{6-2}\int_2^6 \left(-0.5t^2 + 5t + 72.5\right)\,dt = \frac{1}{4}\left[-\frac{0.5}{3}t^3 + \frac{5}{2}t^2 + 72.5t\Big|_{t=2}^{t=6}\right] = \dots = \frac{503}{6} = 83.8\overline{3}$$

   The average temperature between 8 AM and noon is approximately 84°F.

2. Show that $\overline{T}$ acts as true average by showing $\displaystyle\int_2^6 \left(T(t) - \overline{T}\right)\,dt = 0$.

$$\int_2^6 \left(T(t) - \overline{T}\right)\,dt = \int_2^6 \left(-0.5t^2 + 5t + 72.5 - \frac{503}{6}\right)\,dt = \dots = -\frac{1}{6}t^3 + \frac{5}{2}t^2 - \frac{34}{3}t\Big|_{t=2}^{t=6} = (-14) - (-14) = 0 \checkmark$$

3. Does the temperature ever reach the average between 8 AM and noon?

   We set $T(t) = \overline{T}$ so we solve: $-0.5t^2 + 5t + 72.5 = \frac{503}{6}$.

   Clearing fractions, we get $-3t^2 + 30t + 435 = 503$ or $3t^2 - 30t + 68 = 0$.

   The quadratic formula gives $t = \dfrac{15 \pm \sqrt{21}}{3}$. Only $t = \dfrac{15 - \sqrt{21}}{3} \approx 3.47$ lies in the interval $(2, 6)$.

   This means the average temperature is reached at approximately 9:30 AM.

The last two problems showcase two properties of the average value of a function which are true in general.

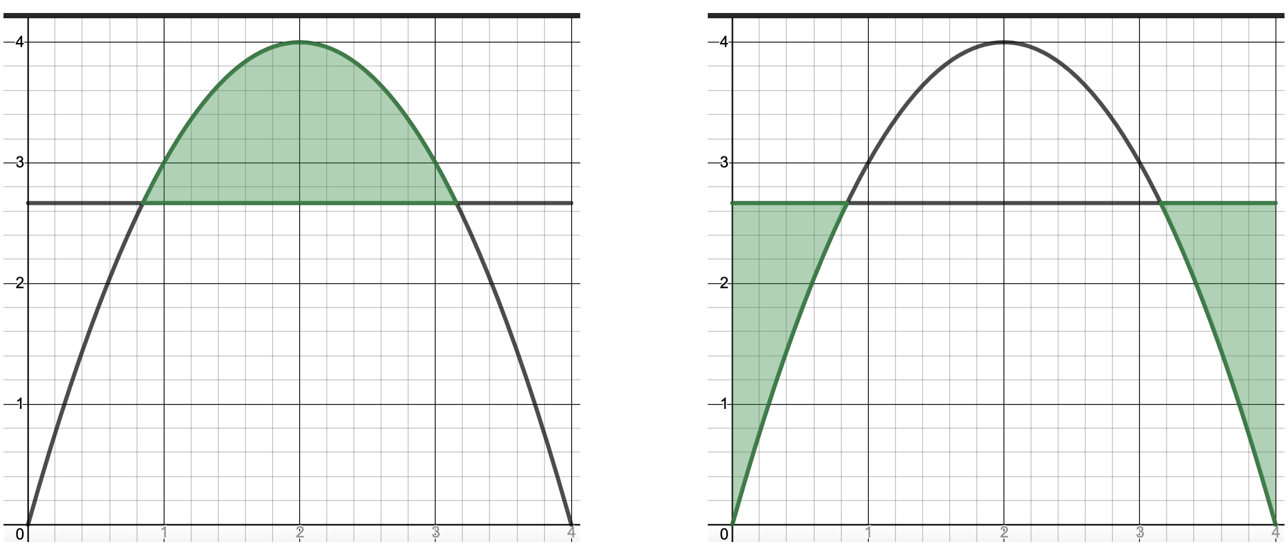
THEOREM: Let $\overline{f} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. Then $\displaystyle\int_a^b \left(f(x) - \overline{f}\right)\,dx = 0$.

PROOF:

$$\begin{aligned}
\int_a^b \left(f(x) - \overline{f}\right)\,dx &= \int_a^b f(x)\,dx - \int_a^b \overline{f}\,dx \\
&= \int_a^b f(x)\,dx - \overline{f}\int_a^b \,dx \\
&= \int_a^b f(x)\,dx - \overline{f}\left(x\Big|_{x=a}^{x=b}\right) \\
&= \int_a^b f(x)\,dx - (b-a)\overline{f} \\
&= \int_a^b f(x)\,dx - (b-a)\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \\
&= \int_a^b f(x)\,dx - \int_a^b f(x)\,dx = 0 \checkmark
\end{aligned}$$

Geometrically, this says the area above the horizontal line $y = \overline{f}$ but below the graph of $y = f(x)$ is the same as the area below the horizontal line $y = \overline{f}$ but above the graph of $y = f(x)$.

Below, the shaded area below on the left is equal to the sum of the shaded areas below on the right.



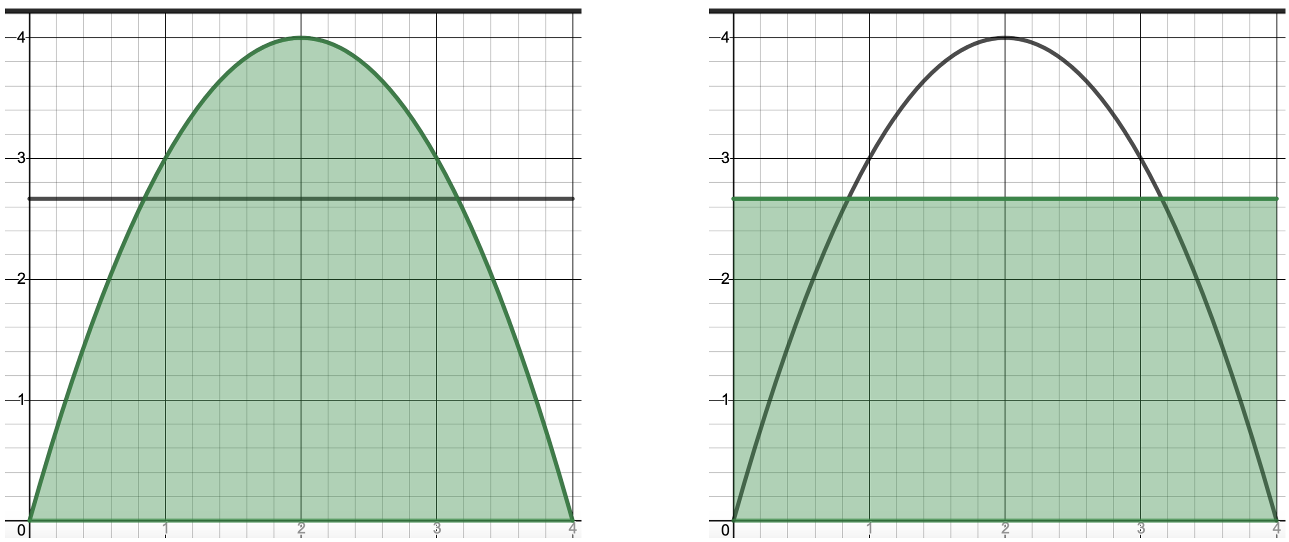
The graph of $y = f(x)$ (the parabola) along with the graph of $y = \overline{f}$ (the horizontal line.)

MEAN VALUE THEOREM FOR DEFINITE INTEGRALS:

If f is continuous on $[a, b]$, then there is a value c in $[a, b]$ with $f(x) = \overline{f}$. That is:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{or, equivalently,} \quad \int_a^b f(x)\,dx = f(c)(b-a)$$

Geometrically, if $f(x) \geq 0$ on $[a, b]$, the MVT for Definite Integrals says we can find at least one value c where the rectangle of width $(b-a)$ and height $f(c)$ contains the same area as the area between the graph of $y = f(x)$ and the x-axis. Below, the shaded area on the left is equal to the shaded area on the right.



The graph of $y = f(x)$ (the parabola) along with the graph of $y = \overline{f}$ (the horizontal line.)

The proof of the Mean Value Theorem of Integrals cites the following results:

RECALL:

- **THE INTERMEDIATE VALUE THEOREM (IVT):**

  If f is continuous on an interval containing real numbers a and b and d is any real number between $f(a)$ and $f(b)$, then there is at least one real number c between a and b with $f(c) = d$.

- **EXTREME VALUE THEOREM:** (EVT) If f is **continuous** on $[a, b]$, then f attains its extrema on $[a, b]$. That is, there are values x_1 and x_2 in $[a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all x in $[a, b]$.

- **PRESERVATION OF ORDER:** Suppose f and g are continuous on $[a, b]$.

  - If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$.
  - More generally, if If $f(x) \geq g(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

PROOF OF THE MVT FOR INTEGRALS: (VIDEO)

Suppose f is continuous on $[a, b]$.

By the EVT, there are x_1 and x_2 in $[a, b]$ with $f(x_1) \leq f(x) \leq f(x_2)$ for all x in $[a, b]$. Hence:

$$\int_a^b f(x_1)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b f(x_2)\,dx$$

Since $f(x_1)$ is a constant, $\displaystyle\int_a^b f(x_1)\,dx = f(x_1)\int_a^b dx = f(x_1)\left(x\Big|_{x=a}^{x=b}\right) = f(x_1)(b-a)$.

Likewise, since $f(x_2)$ is a constant, $\displaystyle\int_a^b f(x_2)\,dx = f(x_2)\int_a^b dx = f(x_2)\left(x\Big|_{x=a}^{x=b}\right) = f(x_2)(b-a)$.

Hence,

$$f(x_1)(b-a) \leq \int_a^b f(x)\,dx \leq f(x_2)(b-a) \quad \text{or} \quad f(x_1) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(x_2)$$

Since f is continuous, the IVT guarantees a c between x_1 and x_2 (possibly x_1 or x_2) with $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ ✓.

NOTE: Unlike the average of a finite set of numbers, which may or may not be among the data set, the Mean Value Theorem for Definite Integrals guarantees that continuous functions attain their average somewhere.

EXAMPLE 3: (VIDEO) Let $f(x) = 3 - x$.

1. Explain why f satisfies the conditions for the Mean Value Theorem for Definite Integrals on $[-1, 3]$.

2. Find all values 'c' guaranteed by the MVT for Definite Integrals. Ans: $c = 1$

3. Check your answer geometrically.

The MVT of Definite Integrals allows us to tighten up our proof of the Fundamental Theorem of Calculus!

MORE FORMAL PROOF OF THE FToC (INTEGRALS ARE ANTIDERIVATIVES): (VIDEO)

Suppose f is continuous on $[a, b]$ and let $F(x) = \displaystyle\int_a^x f(t)\,dt$ for x in (a, b). Recall the definition of derivative:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\displaystyle\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

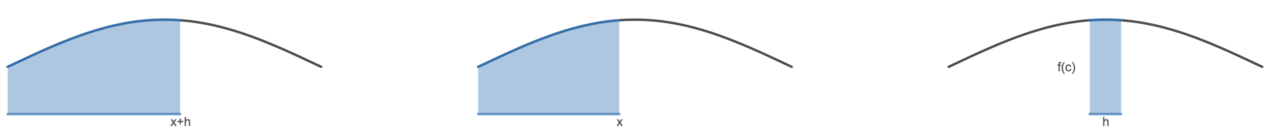
Focusing on the numerator, we use properties of the definite integral to get:

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt = \int_x^a f(t)\,dt + \int_a^{x+h} f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

Applying the MVT for Integrals, there is a value c between x and $x + h$ with

$\displaystyle\int_x^{x+h} f(t)\,dt = f(c)\,(x + h - x) = f(c)\,h.$

Geometrically:



$$\int_a^{x+h} f(t)\,dt \qquad - \qquad \int_a^x f(t)\,dt \qquad = \qquad \int_x^{x+h} f(t)\,dt = f(c)\,h$$

Hence,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\displaystyle\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0} \frac{f(c)\,h}{h} = \lim_{h\to 0} f(c)$$

Since c is between x and $x + h$, as $h \to 0$, $c \to x$. Since f is continuous, $\displaystyle\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$. ✓

HOMEWORK: Section 5.4: 1 - 53 odd, 57*, 59*